**ABE 307**

**Steady Flow in a Long Circular Tube Using Navier Stoke’s Equation**

Recall the problem of steady state flow in a circular pipe. Obtain the velocity profile equation using the Equation of Continuity and Equation of Motion. Assume constant ρ and μ. Write all the assumptions and conditions for the physical situation to be used.

**Assumptions:**

Steady state

No radial flow, vr = 0

No tangential velocity, v𝛳 = 0

vz ≠ 0

Constant density flow (𝛿p/𝛿t) = 0

**Equation of Continuity**

Cylindrical coordinates (r, 𝛳, z):

𝛿ρ/𝛿r + 1/r \* 𝛿/𝛿r (ρrvr) + 1/r \* 𝛿/𝛿𝛳 (ρv𝛳) + 𝛿/𝛿z(ρvz) = 0

~~𝛿ρ/𝛿r~~ + ~~1/r \* 𝛿/𝛿r (ρrv~~~~r~~~~)~~ + ~~1/r \* 𝛿/𝛿𝛳 (ρv~~~~𝛳~~~~)~~ + 𝛿/𝛿z(ρvz) = 0

𝛿/𝛿z(ρvz) = 0 --> vz is not a function of z but a function of r: vz(r)

**Equation of Motion**

R-component:

ρ(∂vr/∂t + vr \* ∂vr/∂r + vθ/r \* ∂vr/∂θ - vθ2/r + vz \* ∂vr/∂z) = -∂p/∂r + μ(∂/∂r \* (1/r \* ∂/∂r \* rvr) + 1/r2 \* ∂2vr/∂θ2 - 2/r2 \* ∂vθ/∂θ) + ρgr

ρ~~(∂v~~~~r~~~~/∂t~~ + ~~v~~~~r~~ ~~\* ∂v~~~~r~~~~/∂r~~ + ~~v~~~~θ~~~~/r \* ∂v~~~~r~~~~/∂θ~~ - ~~v~~~~θ~~~~2~~~~/r~~ + ~~v~~~~z~~ ~~\* ∂v~~~~r~~~~/∂z~~) = -∂p/∂r + μ(~~∂/∂r \* (1/r \* ∂/∂r \* rv~~~~r~~~~)~~ + ~~1/r~~~~2~~ ~~\* ∂~~~~2~~~~v~~~~r~~~~/∂θ~~~~2~~ - ~~2/r~~~~2~~ ~~\* ∂v~~~~θ~~~~/~~~~∂θ~~) + ~~ρg~~~~r~~

-𝛿p/𝛿r = 0

>>> P ≠ P(r) (pressure not variant in radial direction)

𝛳-component:

ρ(∂vθ/∂t + vr \* ∂vθ/∂r + vθ/r \* ∂vθ/∂θ - vθvr/r + vz \* ∂vθ/∂z) = -1/r \* ∂p/∂θ + μ(∂/∂r \* (1/r \* ∂/∂r \* rvθ) + 1/r2 \* ∂2vθ/∂θ2 + 2/r2 \* ∂vr/∂θ + ∂2vθ/∂z2) + ρgθ

ρ(~~∂v~~~~θ~~~~/∂t~~ + ~~v~~~~r~~ ~~\* ∂v~~~~θ~~~~/∂r~~ + ~~v~~~~θ~~~~/r \* ∂v~~~~θ~~~~/∂θ~~ - ~~v~~~~θ~~~~v~~~~r~~~~/r~~ + ~~v~~~~z~~ ~~\* ∂v~~~~θ~~~~/∂z~~) = -1/r \* ∂p/∂θ + μ(~~∂/∂r \* (1/r \* ∂/∂r \* rv~~~~θ~~~~)~~ + ~~1/r~~~~2~~ ~~\* ∂~~~~2~~~~v~~~~θ~~~~/∂θ~~~~2~~+ ~~2/r~~~~2~~ ~~\* ∂v~~~~r~~~~/∂θ~~ + ~~∂~~~~2~~~~v~~~~θ~~~~/∂z~~~~2~~) + ~~ρg~~~~θ~~

-1/r \* 𝛿p/𝛿𝛳 = 0

>>> P ≠ P(𝛳) (pressure not variant in angular direction)

>>> P = f(z)

Z-component:

ρ(∂vz/∂t + vr \* ∂vz/∂r + vθ/r \* ∂vz/∂θ + vz \* ∂vz/∂z) = -∂p/∂z + μ(∂/∂r \* (1/r \* ∂vz/∂r \* r) + 1/r2 \* ∂2vz/∂θ2 + ∂2vz/∂z2) + ρgz

ρ(~~∂v~~~~z~~~~/∂t~~ + ~~v~~~~r~~ ~~\* ∂v~~~~z~~~~/∂r~~ + ~~v~~~~θ~~~~/r \* ∂v~~~~z~~~~/∂θ~~ + ~~v~~~~z~~ ~~\* ∂v~~~~z~~~~/∂z~~) = -∂p/∂z + μ(∂/∂r \* (1/r \* ∂vz/∂r \* r) + ~~1/r~~~~2~~ ~~\* ∂~~~~2~~~~v~~~~z~~~~/∂θ~~~~2~~ + ~~∂~~~~2~~~~v~~~~z~~~~/∂z~~~~2~~) + ρgz

0 = -𝛿p/𝛿z + μ[1/r \* 𝛿/𝛿r(r \* 𝛿vz/𝛿r)] + ρgz

We solve z-component to get velocity profile. Since P & G are dependent on f(y), z and fz is a function of r, the two terms need to be equal to a constant. f(x) + f(y) = 0

0 = -𝛿P/𝛿z + μ[1/r \* 𝛿/𝛿r(r \* 𝛿vz/𝛿r)]

+𝛿P/𝛿z = C0

P = C0z + c1

μ[1/r \* 𝛿/𝛿r(r \* 𝛿vz/𝛿r)] = c0

𝛿/𝛿r(r \* 𝛿vz/𝛿r) = Co r/μ

(r \* 𝛿vz/𝛿r) = Co r2/2μ + C’1

𝛿vz/𝛿r = C0r/2μ + C’1/r

Vz = C0r2/4μ + C’1 \* ln(r) + C2

P = C0z + C1

At z = 0, P = P0

At z = L, P = PL

C1 = P0

PL = C0L + C1

C0 = PL - C1 / L = PL - P0 / L

P = (PL - P0)z/L + P0

Vz = C0r2/4μ + C2 (at r = 0, we need velocity to be finite)

At r = R, vz = 0

0 = C0R2/4μ + C2

C2 = C0R2/4μ

vz = (PL-P0)/L \* r2/4μ - (PL-P0)/L \* R2/4μ

Vz = (PL - P0)/4μL [r2 - R2]